

Controlling a resonant transmission across the δ' -potential: the inverse problem

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Abstract. Recently, the non-zero transmission of a quantum particle through the one-dimensional singular potential given in the form of the derivative of Dirac's delta function, $\lambda\delta'(x)$, with $\lambda \in \mathbb{R}$, being a potential strength constant, has been discussed by several authors. The transmission occurs at certain discrete values of λ forming a resonance set $\{\lambda_n\}_{n=1}^{\infty}$. For $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$ this potential has been shown to be a perfectly reflecting wall. However, this resonant transmission takes place only in the case when the regularization of the distribution $\delta'(x)$ is constructed in a specific way. Otherwise, the δ' -potential is fully non-transparent. Moreover, when the transmission is non-zero, the structure of a resonant set depends on a regularizing sequence $\Delta'_\varepsilon(x)$ that tends to $\delta'(x)$ in the sense of distributions as $\varepsilon \rightarrow 0$. Therefore, from a practical point of view, it would be interesting to have an inverse solution, i.e. for a given $\bar{\lambda} \in \mathbb{R}$ to construct such a regularizing sequence $\Delta'_\varepsilon(x)$ that the δ' -potential at this value is transparent. If such a procedure is possible, then this value $\bar{\lambda}$ has to belong to a corresponding resonance set. The present paper is devoted to solving this problem and, as a result, the family of regularizing sequences is constructed by tuning adjustable parameters in the equations that provide a resonance transmission across the δ' -potential. This construction can be realized if each regularizing sequence $\Delta'_\varepsilon(x)$ depends on $\lambda \in \mathbb{R}$ and this is a key point of our approach. Next, we can solve the inverse problem if the regularization is constructed from rectangles. Since in some cases the renormalization procedure $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ leads to the existence of an effective δ -interaction, it is reasonable from the beginning to consider the linear combination $V(x) = \eta\delta(x) + \lambda\delta'(x)$ with $(\eta, \lambda) \in \mathbb{R}^2$.

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1. Introduction

The Schrödinger operators with singular zero-range potentials attract a considerable interest beginning from the pioneering work of Berezin and Faddeev [1]. These operators (for details and references see book [2]) describe point or contact interactions which are widely used in various applications to quantum physics [3, 4, 5, 6, 7, 8]. Intuitively, these interactions are understood as sharply localized potentials, exhibiting a number of interesting and intriguing features. Applications of these models to condensed matter physics (see, e.g., [9, 10, 11, 12]) are of particular interest nowadays, mainly because of the rapid progress in fabricating nanoscale quantum devices, particularly, thin quantum waveguides [13, 14].

In this paper, we consider the one-dimensional Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \quad (1)$$

where the prime stands for the differentiation with respect to the spatial coordinate x and $\psi(x)$ is the wavefunction for a particle of mass m and energy E (we use units in which $\hbar^2/2m = 1$). Using the regularization

$$\varepsilon^{-2}\mathcal{V}(x/\varepsilon) \rightarrow \delta'(x) \quad \text{as } \varepsilon \rightarrow 0 \quad (2)$$

(in the sense of distributions), Šeba[15] has rigorously studied equation (1) with the potential $V(x) = \lambda\delta'(x)$, where $\lambda \in \mathbb{R}$ is an interaction strength constant and $\delta'(x)$ is the derivative of Dirac's delta function $\delta(x)$. Here, $\mathcal{V}(\xi)$, $\xi \in \mathbb{R}$, is assumed to be a smooth compactly supported function satisfying the conditions

$$\int_{\mathbb{R}} \mathcal{V}(\xi)d\xi = 0 \quad \text{and} \quad \int_{\mathbb{R}} \xi\mathcal{V}(\xi)d\xi = -1. \quad (3)$$

As a result, he has obtained in the $\varepsilon \rightarrow 0$ limit the direct sum of the free Schrödinger operators $-d^2/dx^2$ on the negative and positive half-axes of \mathbb{R} with Dirichlet boundary conditions at the origin $x = 0$. In physical terms this means that the δ' -barrier is completely nontransparent for quantum particles. The absence of a nontrivial point interaction in the zero-range limit has prompted Šeba [15] to introduce and analyze a less singular interaction (with a renormalized interaction strength) through the limit

$$V(\alpha; x) = \lambda \lim_{\varepsilon \rightarrow 0} \frac{\delta(x + \varepsilon) - \delta(x - \varepsilon)}{2\varepsilon^\alpha} \quad (4)$$

with $\alpha \in (0, 1]$. In the limiting case $\alpha = 1$, limit (4) gives the unrenormalized point dipole interaction $\lambda\delta'(x)$. For this interaction he has proved that for $\alpha < 1/2$ limit (4) is trivial, i.e. the system behaves as if the point potential is absent, while for $\alpha > 1/2$ the system splits into two independent subsystems separated on the half-axes $(-\infty, 0)$ and $(0, \infty)$. The only non-trivial case has been proved to occur for $\alpha = 1/2$, when the point interaction appears to be $V(1/2; x) = g\delta(x)$ with the effective coupling constant $g = -\lambda^2/2$. Recently, Šeba's approach has been generalized from one to two [16] and three [17] dimensions. More precisely, instead of the one power α in (4), the regularization procedure using three powers μ, ν, τ has been developed. As a result, a

three-dimensional manifold has been constructed in the $\{\mu, \nu, \tau\}$ -space that corresponds to the δ -interaction with different values of the effective coupling constant g .

Afterward there were other attempts to regularize the δ' -potential using both local and nonlocal regular functions. However, the calculations of scattering amplitudes in [18] on a piecewise δ' -like approximating potentials have discovered that at certain discrete values of λ the transmission across the δ' -barrier is non-zero. These calculations contradict Šeba's result for $\alpha = 1$ and later on Golovaty and Man'ko[19] have rigorously proved the existence of non-zero transmission for a wide class of δ' -like regularizing sequences with compact supports. This discrepancy has prompted Golovaty and Hryniv[20] to revise Šeba's result. They have proved that for $\lambda\delta'$ -like potentials with compact supports there exists a sufficiently large set of the coupling constant λ at which the norm convergent limit differs from the operator obtained by Šeba.

In this paper, we develop a regularizing procedure which allows us to get a transparent regime for a given value of λ . To this end, for any $\mu > 1$, instead of (2), we approximate the singular potential $V(x) = \lambda\delta'(x)$ by

$$V_\varepsilon(\lambda; x) = \lambda\varepsilon^{2(1-\mu)}\mathcal{V}_\varepsilon(\lambda; \varepsilon^{(1-\mu)}x) \rightarrow \lambda\delta'(x) \quad (5)$$

where the sequence $\mathcal{V}_\varepsilon(\lambda; \xi)$ depends in general on λ and instead of both conditions (3) we require the limiting equality

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \xi \mathcal{V}_\varepsilon(\lambda; \xi) d\xi = -1. \quad (6)$$

As shown previously [16, 17], in some cases the barrier-well regularizing sequence leads in the zero-range limit to an additional effective δ -interaction. Therefore, similarly to [21], it is reasonable to add to the δ' -potential a pure δ -potential, so that the total potential $V(x)$ in this paper is assumed to be the sum of Dirac's delta function and its derivative, i.e.

$$V(x) = \eta\delta(x) + \lambda\delta'(x), \quad (7)$$

where $\eta \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ are strength interaction constants for the δ - and δ' -potentials, respectively.

Since we do not impose the first constraint of (3), we need to define a space of test functions discontinuous at the origin. In this way, we slightly extend the family of point interactions from the standard δ' -potential to a wider class of δ' -like barriers.

2. Definition of δ - and δ' -like distributions on a space of discontinuous test functions

The use of distributions on discontinuous test functions has been considered first by Griffiths[22] and the general theory in this direction has been developed by Kurasov[23]. For other applications of the distributions for discontinuous test functions see, e.g., [21, 24, 25, 26]. Here, we restrict ourselves only to the class of discontinuous functions the all derivatives of which are continuous. The only purpose of this slight

generalization is to avoid the first equality (3) and therefore to keep only the constraint (6).

Let $\varphi(x)$ be a test function from the \mathcal{D} space. Shifting a positive part of this function by a non-zero constant, one can form a space of discontinuous at $x = 0$ test functions. To this end, for a fixed $\varsigma \in \mathbb{R}$ we assume

$$\varphi_\varsigma(x) \doteq \varphi(x) + (\varsigma - 1)\varphi(0)\Theta(x), \quad (8)$$

where $\Theta(x)$ is the Heaviside function. This function is discontinuous at $x = 0$ if $\varsigma \neq 1$, i.e. $\varphi_\varsigma(-0) = \varphi(0)$ and $\varphi_\varsigma(+0) = \varsigma\varphi(0)$, while its two-sided derivatives are continuous at $x = \pm 0$. For each $\varsigma \neq 1$ we denote the space of these test functions by \mathcal{D}_ς . Clearly, there exists a one-to-one correspondence between the spaces \mathcal{D} and \mathcal{D}_ς at a given ς and in the particular case $\varsigma = 1$ the space \mathcal{D}_ς coincides with \mathcal{D} . Due to the boundary conditions at $x = 0$, one can slightly modify, e.g., the distributions $\delta(x)$ and $\delta'(x)$ as follows: $\delta(x) : \varphi_\varsigma(x) \rightarrow \varsigma\varphi(0)$ and $\delta'(x) : \varphi_\varsigma(x) \rightarrow -\varphi'(0)$.

Now our purpose is to construct the regularizing sequences $\Delta_\varepsilon(x)$ and $\Delta'_\varepsilon(x)$ (the prime here does not denote the differentiation) such that $\Delta_\varepsilon(x) \rightarrow \delta(x)$ and $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ in the sense of distributions defined above on the space \mathcal{D}_ς . Introducing the new spatial variable $\xi = \varepsilon^{1-\mu}x$ with $\mu > 1$, we define new regularizing functions $\mathcal{V}_\varepsilon(\xi)$ and $\mathcal{V}'_\varepsilon(\xi)$ through the relations

$$\Delta_\varepsilon(x) = \varepsilon^{1-\mu}\mathcal{V}_\varepsilon(\xi) \quad \text{and} \quad \Delta'_\varepsilon(x) = \varepsilon^{2(1-\mu)}\mathcal{V}'_\varepsilon(\xi). \quad (9)$$

Again, the prime in $\mathcal{V}'_\varepsilon(\xi)$, the same as in $\Delta'_\varepsilon(x)$, does not mean differentiation. Integrating these functions over \mathbb{R}^- and \mathbb{R}^+ separately and expanding $\varphi_\varsigma(\xi)$ from the left and the right of the origin $x = \pm 0$, we obtain the expansions

$$\begin{aligned} \langle \Delta_\varepsilon(x) | \varphi_\varsigma(x) \rangle &\doteq \int_{\mathbb{R}} \Delta_\varepsilon(x) \varphi_\varsigma(x) dx = \int_{\mathbb{R}} \mathcal{V}_\varepsilon(\xi) \varphi_\varsigma(\varepsilon^{\mu-1}\xi) d\xi \\ &= \int_{\mathbb{R}^-} \mathcal{V}_\varepsilon(\xi) [\varphi(0) + \varepsilon^{\mu-1}\xi \varphi'(0) + \dots] d\xi \\ &+ \int_{\mathbb{R}^+} \mathcal{V}_\varepsilon(\xi) [\varsigma\varphi(0) + \varepsilon^{\mu-1}\xi \varphi'(0) + \dots] d\xi \\ &= m_{0,\varsigma}(\varepsilon) \varsigma\varphi(0) + \dots + \varepsilon^{j(\mu-1)} m_j(\varepsilon) \varphi^{(j)}(0) + \dots, \end{aligned} \quad (10)$$

where

$$m_{0,\varsigma}(\varepsilon) \doteq \varsigma^{-1} \int_{\mathbb{R}^-} \mathcal{V}_\varepsilon(\xi) d\xi + \int_{\mathbb{R}^+} \mathcal{V}_\varepsilon(\xi) d\xi, \quad m_j(\varepsilon) \doteq \frac{1}{j!} \int_{\mathbb{R}} \xi^j \mathcal{V}_\varepsilon(\xi) d\xi, \quad (11)$$

with $j = 1, 2, \dots$. As follows from this expansion, for the proper definition of the $\delta(x)$ function, we need to have

$$\lim_{\varepsilon \rightarrow 0} m_{0,\varsigma}(\varepsilon) = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} m_j(\varepsilon) = \text{const} \quad (12)$$

for all $j = 1, 2, \dots$

Similarly, we write

$$\langle \Delta'_\varepsilon(x) | \varphi_\varsigma(x) \rangle \doteq \int_{\mathbb{R}} \Delta'_\varepsilon(x) \varphi_\varsigma(x) dx = \varepsilon^{1-\mu} \int_{\mathbb{R}} \mathcal{V}'_\varepsilon(\xi) \varphi_\varsigma(\varepsilon^{\mu-1}\xi) d\xi$$

$$\begin{aligned}
 &= \varepsilon^{1-\mu} \int_{\mathbb{R}^-} \mathcal{V}'_\varepsilon(\xi) [\varphi(0) + \varepsilon^{\mu-1} \xi \varphi'(0) + \dots] d\xi \\
 &+ \varepsilon^{1-\mu} \int_{\mathbb{R}^+} \mathcal{V}'_\varepsilon(\xi) [\varsigma \varphi(0) + \varepsilon^{\mu-1} \xi \varphi'(0) + \dots] d\xi \\
 &= \varepsilon^{1-\mu} m'_{0,\varsigma}(\varepsilon) \varsigma \varphi(0) + \dots + \varepsilon^{(j-1)(\mu-1)} m'_j(\varepsilon) \varphi^{(j)}(0) + \dots,
 \end{aligned} \tag{13}$$

where

$$m'_{0,\varsigma}(\varepsilon) \doteq \varsigma^{-1} \int_{\mathbb{R}^-} \mathcal{V}'_\varepsilon(\xi) d\xi + \int_{\mathbb{R}^+} \mathcal{V}'_\varepsilon(\xi) d\xi, \quad m'_j(\varepsilon) \doteq \frac{1}{j!} \int_{\mathbb{R}} \xi^j \mathcal{V}'_\varepsilon(\xi) d\xi, \tag{14}$$

with $j = 1, 2, \dots$. Here, we have to examine the following two possibilities: $\Delta'_\varepsilon(x) \rightarrow \delta(x)$ and $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ on the space of discontinuous test functions from the \mathcal{D}_ς space. In the former case, we have to satisfy the equations

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1-\mu} m'_{0,\varsigma}(\varepsilon) = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} m'_j(\varepsilon) = 0 \tag{15}$$

for all $j = 1, 2, \dots$, whereas in the latter case, we need to have

$$m'_{0,\varsigma}(\varepsilon) = 0 \quad \text{for any } \varepsilon > 0, \quad \lim_{\varepsilon \rightarrow 0} m'_1(\varepsilon) = -1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} m'_j(\varepsilon) = \text{const} \tag{16}$$

for all $j = 2, 3, \dots$

3. A rectangular model and its power parametrization

In this paper, we construct a regularizing sequence for the singular potential (7) consisting of three adjacent rectangular barriers/wells. More precisely, the regularizing sequences $\Delta_\varepsilon(x)$ and $\Delta'_\varepsilon(x)$ for the rectangular model are specified through piecewise functions as follows:

$$\Delta_\varepsilon(x) = \begin{cases} h & \text{for } 0 < x < \rho, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Delta'_\varepsilon(x) = \begin{cases} h_1 & \text{for } -l < x < 0, \\ h_2 & \text{for } \rho < x < \rho + r, \\ h_3 & \text{for } 0 < x < \rho, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

with the constraint $h_1 h_2 < 0$ (double-well structure). We parametrize the rectangular parameters $h, h_1, h_2, h_3, l, \rho, r$ by powers as

$$\begin{aligned}
 h_1 &= a_1 \varepsilon^{2(1-\mu)} + \frac{c_0}{c_1} \varepsilon^{-\mu}, \quad h_2 = a_2 \varepsilon^{2(1-\mu)} - \frac{c_0}{c_2} \varepsilon^{-\nu}, \quad h_3 = a_3 \varepsilon^{2(1-\mu)}, \\
 h &= c_3^{-1} \varepsilon^{-\tau}, \quad l = c_1 \varepsilon, \quad r = \frac{c_2}{\varsigma} \varepsilon^{1-\mu+\nu}, \quad \rho = c_3 \varepsilon^\tau,
 \end{aligned} \tag{18}$$

where a_i ($i = 1, 2, 3$), c_j ($j = 0, 1, 2, 3$), ς are positive numbers, and ε is a squeezing parameter.

Inserting parametrization (18) into (17), according to definition (9), one can write

$$\mathcal{V}_\varepsilon(\xi) = \begin{cases} c_3^{-1} \varepsilon^{\mu-1-\tau} & \text{for } 0 < \xi < c_3 \varepsilon^{1-\mu+\tau}, \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

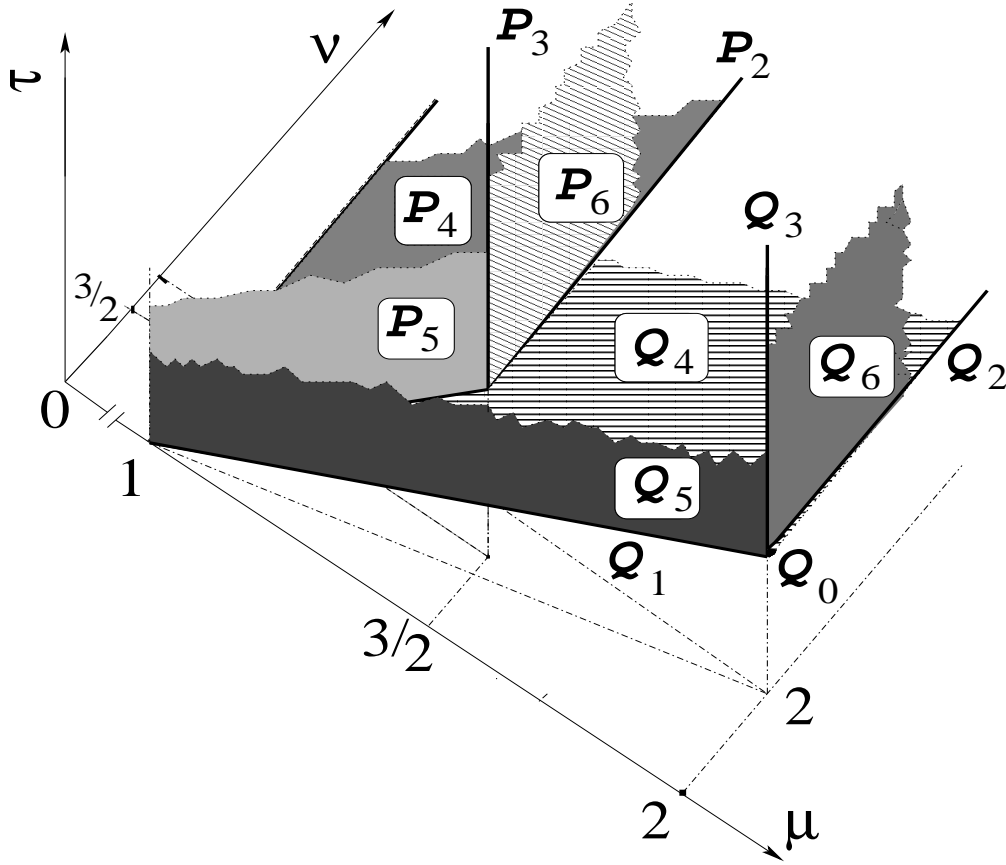


Figure 1. Trihedral surfaces S_δ and $S_{\delta'}$ defined through limits (21) and (22). The surface S_δ is formed by apex P_0 , edges P_1 , P_2 , P_3 and planes P_4 , P_5 , P_6 . Similarly, the surface $S_{\delta'}$ is formed by apex Q_0 , edges Q_1 , Q_2 , Q_3 and planes Q_4 , Q_5 , Q_6 . Notations for apex P_0 and edge P_1 are omitted.

and

$$\mathcal{V}'_\varepsilon(\xi) = \begin{cases} a_1 + (c_0/c_1)\varepsilon^{\mu-2} & \text{for } -c_1\varepsilon^{2-\mu} < \xi < 0, \\ a_2 - (c_0/c_2)\varepsilon^{2\mu-2-\nu} & \text{for } c_3\varepsilon^{1-\mu+\tau} < \xi < (c_2/\varsigma)\varepsilon^{2-2\mu+\nu} \\ & + c_3\varepsilon^{1-\mu+\tau}, \\ a_3 & \text{for } 0 < \xi < c_3\varepsilon^{1-\mu+\tau}, \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Obviously, for any $\varepsilon > 0$ we have $m_{0,\varsigma}(\varepsilon) = 1$ and the second limit (12) is easily proved by induction for any positive c_3 if $\tau \geq \mu - 1$. Therefore, in this case $\Delta_\varepsilon(x) \rightarrow \delta(x)$ in the sense of the \mathcal{D}' -distributions. As mentioned above, one can consider separately the δ - and δ' -limits of $\Delta'_\varepsilon(x)$ and this depends on the choice of the constants a_1, a_2, a_3 . Indeed, if these constants are non-zero, the first limit (15) leads to

$$\lim_{\varepsilon \rightarrow 0} \left[\varepsilon^{2(1-\mu)} \left(\frac{a_1 c_1}{\varsigma} \varepsilon + \frac{a_2 c_2}{\varsigma} \varepsilon^{1-\mu+\nu} + a_3 c_3 \varepsilon^\tau \right) \right] = 1. \quad (21)$$

This limit determines the trihedral surface S_δ (see Fig. 1) being a subset of the $\{\mu, \nu, \tau\}$ -space with $\nu, \tau \geq \mu - 1$. On this surface, the second zero limit (15) holds and, using

Table 1. Definition of apex P_0 , edges P_1, P_2, P_3 and planes P_4, P_5, P_6 forming the trihedral surface S_δ together with the constraints on constants $a_1, a_2, a_3; c_1, c_2, c_3; \varsigma$, satisfying the limit (21).

P_j -sets	Constraints on $a_1, a_2, a_3; c_1, c_2, c_3; \varsigma$
$P_0 \doteq \{\mu = \nu = 3/2, \tau = 1\}$	$a_1 c_1 + a_2 c_2 + a_3 c_3 \varsigma = \varsigma$
$P_1 \doteq \{1 < \mu < 3/2, \nu = 3(\mu - 1), \tau = 2(\mu - 1)\}$	$a_2 c_2 + a_3 c_3 \varsigma = \varsigma$
$P_2 \doteq \{\mu = 3/2, \nu > 3/2, \tau = 1\}$	$a_1 c_1 + a_3 c_3 \varsigma = \varsigma$
$P_3 \doteq \{\mu = \nu = 3/2, \tau > 1\}$	$a_1 c_1 + a_2 c_2 = \varsigma$
$P_4 \doteq \{1 < \mu < 3/2, \nu > 3(\mu - 1), \tau = 2(\mu - 1)\}$	$a_3 c_3 = 1$
$P_5 \doteq \{1 < \mu < 3/2, \nu = 3(\mu - 1), \tau > 2(\mu - 1)\}$	$a_2 c_2 = \varsigma$
$P_6 \doteq \{\mu = 3/2, \nu > 3/2, \tau > 1\}$	$a_1 c_1 = \varsigma$

Table 2. Definition of apex Q_0 , edges Q_1, Q_2, Q_3 and planes Q_4, Q_5, Q_6 forming the trihedral surface $S_{\delta'}$ together with the constraints on constants $c_0, c_1, c_2, c_3; \varsigma$, satisfying limit (22).

Q_j -sets	Constraints on $c_0, c_1, c_2, c_3; \varsigma$
$Q_0 \doteq \{\mu = \nu = 2, \tau = 1\}$	$\frac{1}{2}(c_1 + c_2 \varsigma^{-2}) + \frac{c_3}{\varsigma} = \frac{1}{c_0}$
$Q_1 \doteq \{1 < \mu < 2, \nu = 2(\mu - 1), \tau = \mu - 1\}$	$\frac{c_2}{2\varsigma} + c_3 = \frac{\varsigma}{c_0}$
$Q_2 \doteq \{\mu = 2, \nu > 2, \tau = 1\}$	$\frac{c_1}{2} + \frac{c_3}{\varsigma} = \frac{1}{c_0}$
$Q_3 \doteq \{\mu = \nu = 2, \tau > 1\}$	$c_1 + c_2 \varsigma^{-2} = \frac{2}{c_0}$
$Q_4 \doteq \{1 < \mu < 2, \nu > 2(\mu - 1), \tau = \mu - 1\}$	$c_0 c_3 = \varsigma$
$Q_5 \doteq \{1 < \mu < 2, \nu = 2(\mu - 1), \tau > \mu - 1\}$	$c_0 c_2 = 2\varsigma^2$
$Q_6 \doteq \{\mu = 2, \nu > 2, \tau > 1\}$	$c_0 c_1 = 2$

direct calculations, it is proved by induction. The S_δ surface is formed by the apex P_0 , the edges P_1, P_2, P_3 , and the planes P_4, P_5, P_6 , defined in table 1. Limiting equation (21) results in the constraints on the constants $a_1, a_2, a_3; c_1, c_2, c_3; \varsigma$, depending on the sets P_j , $j = 0, 1, \dots, 6$. These constraints are summarized in table 1.

Obviously, the first condition (16) is fulfilled if $a_1 = a_2 = a_3 = 0$. Calculating the first moment $m'_1(\varepsilon)$, one finds that the second limit (16) leads to the condition

$$\lim_{\varepsilon \rightarrow 0} \left\{ \varepsilon^{1-\mu} c_0 \left[\frac{1}{2} \left(c_1 \varepsilon + \frac{c_2}{\varsigma^2} \varepsilon^{1-\mu+\nu} \right) + \frac{c_3}{\varsigma} \varepsilon^\tau \right] \right\} = 1. \quad (22)$$

Using this limit, the third condition (16) can easily be established by induction. Limiting equation (22) determines another trihedral surface $S_{\delta'}$ (see figure 1), which corresponds to the δ' -limit. This surface is formed by the apex Q_0 , the edges Q_1, Q_2, Q_3 and the planes Q_4, Q_5, Q_6 defined in table 2. Condition (22) also imposes the constraints on the constants c_0, c_1, c_2, c_3 and ς summarized in table 2.

4. A finite-range solution for the rectangular model

The solution of equation (1) can be written through the transfer matrix Λ connecting the boundary conditions for the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ at $x = x_1 = -l$ and $x = x_2 = \rho + r$:

$$\begin{pmatrix} \psi(x_2) \\ \psi'(x_2) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(x_1) \\ \psi'(x_1) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}. \quad (23)$$

As a result, we obtain

$$\begin{aligned} \Lambda_{11} &= \left[\cos(pl) \cos(qr) - \frac{p}{q} \sin(pl) \sin(qr) \right] \cos(s\rho) \\ &\quad - \left[\frac{p}{s} \sin(pl) \cos(qr) + \frac{s}{q} \cos(pl) \sin(qr) \right] \sin(s\rho), \\ \Lambda_{12} &= \left[\frac{1}{p} \sin(pl) \cos(qr) + \frac{1}{q} \cos(pl) \sin(qr) \right] \cos(s\rho) \\ &\quad + \left[\frac{1}{s} \cos(pl) \cos(qr) - \frac{s}{pq} \sin(pl) \sin(qr) \right] \sin(s\rho), \\ \Lambda_{21} &= - \left[p \sin(pl) \cos(qr) + q \cos(pl) \sin(qr) \right] \cos(s\rho) \\ &\quad - \left[s \cos(pl) \cos(qr) - \frac{pq}{s} \sin(pl) \sin(qr) \right] \sin(s\rho), \\ \Lambda_{22} &= \left[\cos(pl) \cos(qr) - \frac{q}{p} \sin(pl) \sin(qr) \right] \cos(s\rho) \\ &\quad - \left[\frac{s}{p} \sin(pl) \cos(qr) + \frac{q}{s} \cos(pl) \sin(qr) \right] \sin(s\rho), \end{aligned} \quad (24)$$

where the quantities

$$p \doteq \sqrt{E - \lambda h_1}, \quad q \doteq \sqrt{E - \lambda h_2}, \quad s \doteq \sqrt{E - \eta h - \lambda h_3} \quad (25)$$

can be either real or imaginary.

5. Asymptotical analysis: basic expansions

The parametrization given by equations (18) is a key point in our approach. Inserting these equations into (25), in the $\varepsilon \rightarrow 0$ limit one can write the following asymptotics:

$$\begin{aligned} p &\rightarrow \varepsilon^{1-\mu} \sqrt{-\lambda \left(a_1 + \frac{c_0}{c_1} \varepsilon^{\mu-2} \right)}, \quad q \rightarrow \varepsilon^{1-\mu} \sqrt{-\lambda \left(a_2 - \frac{c_0}{c_2} \varepsilon^{2\mu-2-\nu} \right)}, \\ s &\rightarrow \varepsilon^{1-\mu} \sqrt{-\lambda a_3 - (\eta/c_3) \varepsilon^{2\mu-2-\tau}}. \end{aligned} \quad (26)$$

Expanding the sin- and cos-expressions in (24) up to the second order and using the asymptotics (26) together with equations (18), one can calculate the asymptotics of the matrix elements $\Lambda_{ij} = \Lambda_{ij}(\varepsilon)$, $i, j = 1, 2$, as $\varepsilon \rightarrow 0$. The element Λ_{21} appears to be the most singular term in the region $\mu > 1$ as $\varepsilon \rightarrow 0$. Both on the S_δ and $S_{\delta'}$ surfaces, it can be well defined only if an appropriate cancellation of singularities occurs in the $\varepsilon \rightarrow 0$ limit. Therefore, we start with the analysis of the expansion for this element, arranging

Table 3. Possible zero-range limits of pl and qr in sin- and cos-expressions of transfer matrix solution (24) resulting in constraints on μ and ν , and elements of surface $S_{\delta'}$ where these limits are realized.

	μ	ν	$\tau = \mu - 1$	$\tau \geq 2(\mu - 1)$
(i)	$1 < \mu < 2$	$\nu > 2(\mu - 1)$	Q_4	S_δ
(ii)	$1 < \mu < 2$	$\nu = 2(\mu - 1)$	Q_1	Q_5
(iii)	$\mu = 2$	$\nu > 2$	Q_2	Q_6
(iv)	$\mu = 2$	$\nu = 2$	Q_0	Q_3

the terms (given in powers of ε) in the series $\Lambda_{21} = \Lambda_{21}^{(0)} + \Lambda_{21}^{(1)} + \dots$, where the group of terms $\Lambda_{21}^{(0)}$ contains divergences which under appropriate constraints cancel out in the $\varepsilon \rightarrow 0$ limit. Under these constraints appearing on $S_{\delta'}$ in the form of transcendental equations (called hereafter *transparency equations*), a non-zero transmission across the limiting zero-range potential $V(x)$ occurs. The next group $\Lambda_{21}^{(1)}$ contains the terms which appear to be finite either on the whole S_δ surface or in some non-empty subsets of $S_{\delta'}$ if the transparency equations are taken into account. We denote this limit, which may be either zero or non-zero, as $\lim_{\varepsilon \rightarrow 0} \Lambda_{21}^{(1)} \doteq g$ and the corresponding (transparency) subsets as $T_j \subset Q_j$, $j = 0, 1, \dots, 6$. The next terms of the expansion tend to zero on the sets as $\varepsilon \rightarrow 0$. Using next the transparency equations, the other matrix elements either on S_δ or in T_j 's can be calculated explicitly. As a result, one finds that $\lim_{\varepsilon \rightarrow 0} \Lambda_{12} = 0$, $\lim_{\varepsilon \rightarrow 0} \Lambda_{11} = \chi$ and $\lim_{\varepsilon \rightarrow 0} \Lambda_{22} = \chi^{-1}$ with a finite value χ , so that the connection matrix in all the cases with a non-zero transmission takes the form

$$\Lambda = \begin{pmatrix} \chi & 0 \\ g & \chi^{-1} \end{pmatrix}. \quad (27)$$

Particularly, as shown below, due to the cancellation of divergences (when $\mu > 1$) in the trihedral surrounded by the surface S_δ one obtains $\chi = 1$ and $g = 0$, i.e. the full transmission, while on its boundary $S_{\delta'}$, we have $\chi = 1$ but $g \neq 0$ depending on the element P_j , $j = 0, 1, \dots, 6$. Therefore, in the latter case g may be called the coupling constant of an effective δ -interaction. Outside this trihedral, $g \rightarrow \infty$, i.e. the potential $V(x)$ is fully nontransparent, except for the T_j -sets where $\chi \neq 1$ and g may be non-zero.

Because of the form of equations (24), it is convenient to write the expansion for Λ_{21} in the $\varepsilon \rightarrow 0$ limit separately for the following four cases: (i) $pl \rightarrow 0$ and $qr \rightarrow 0$, (ii) $pl \rightarrow 0$ but qr tends to a non-zero finite constant, (iii) pl goes to a non-zero finite constant while $qr \rightarrow 0$, (iv) both pl and qr tend to non-zero finite constants. As follows from equations (18) and asymptotics (26), each of the four cases leads to certain constraints on μ and ν (see table 3).

We write the expansions in powers of ε for each of cases (i), \dots , (iv), separately, and then perform the explicit cancellation of divergences on each element of the surfaces S_δ and $S_{\delta'}$. As summarized in table 1, some elements of $S_{\delta'}$ lie in the $\tau = \mu - 1$ plane, whereas the others including the surface S_δ belong to the $\tau \geq 2(\mu - 1)$ half-space. In

particular, for the S_δ surface we obtain the following expansion:

$$\begin{aligned} \Lambda_{21} = & \eta + \lambda c_0 \left(1 - \frac{1}{\varsigma}\right) \varepsilon^{1-\mu} + c_1 \left[\lambda a_1 + \frac{\lambda^2 c_0^2}{2} \left(\frac{1}{3} - \frac{1}{\varsigma}\right) \right] \varepsilon^{3-2\mu} \\ & + c_2 \left[\frac{\lambda a_2}{\varsigma} + \frac{\lambda^2 c_0^2}{2\varsigma^2} \left(\frac{1}{3\varsigma} - 1\right) \right] \varepsilon^{3-3\mu+\nu} + c_3 \left(\lambda a_3 - \frac{\lambda^2 c_0^2}{\varsigma} \right) \varepsilon^{2-2\mu+\tau} + \dots \end{aligned} \quad (28)$$

Here, the cancellation of divergences occurs only if $\varsigma = 1$. As a result, the coupling constant of the total δ -interaction becomes

$$\begin{aligned} g = & \eta + \lambda \lim_{\varepsilon \rightarrow 0} \left[c_1 \left(a_1 - \frac{\lambda}{3} c_0^2 \right) \varepsilon^{3-2\mu} + c_2 \left(a_2 - \frac{\lambda}{3} c_0^2 \right) \varepsilon^{3-3\mu+\nu} \right. \\ & \left. + c_3 \left(a_3 - \lambda c_0^2 \right) \varepsilon^{2-2\mu+\tau} \right]. \end{aligned} \quad (29)$$

Finally, using here the limit (21), we obtain

$$g = \eta + \lambda - \frac{\lambda^2 c_0^2}{3} \begin{cases} c_1 + c_2 + 3c_3 & \text{for } P_0, \\ c_2 + 3c_3 & \text{for } P_1, \\ c_1 + 3c_3 & \text{for } P_2, \\ c_1 + c_2 & \text{for } P_3, \\ 3c_3 & \text{for } P_4, \\ c_2 & \text{for } P_5, \\ c_1 & \text{for } P_6. \end{cases} \quad (30)$$

All these constants are positive and arbitrary. The last term that depends on P_j is a renormalization of the coupling constant λ . In the interior of the S_δ surface we have $g = 0$ and outside this surface $g \rightarrow \infty$. Next, as follows from equations (24), $\Lambda_{11}, \Lambda_{22} \rightarrow 1$ as $\varepsilon \rightarrow 0$, so that $\chi = 1$ on the S_δ surface including its interior.

Thus, the cancellation of divergences occurs both in the interior of S_δ (full transmission, $g = 0$) and on its boundary resulting in an effective interaction of the δ -type. Therefore, the S_δ surface serves as a transition region from full to zero transmission.

6. Transparency regimes on the $S_{\delta'}$ surface

Similarly to the expansion (28), the asymptotical analysis on the $S_{\delta'}$ surface has to be performed separately on each element Q_j , $j = 0, 1, \dots, 6$, of this surface. These seven cases with the corresponding constraints on μ and ν are given in table 3. Thus, using equations (18) and asymptotics (26) with $a_1 = a_2 = a_3 = 0$, and expanding the sin- and cos-expressions in equations (24) up to the third order, we obtain the following series for cases (i), \dots , (iv):

$$\begin{aligned} \Lambda_{21}(Q_4) = & \lambda c_0 \left(1 - \frac{1}{\varsigma} - \frac{\lambda c_0 c_3}{\varsigma} \varepsilon^{1-\mu+\tau} \right) \varepsilon^{1-\mu} - \frac{\lambda^2 c_0^2}{2} \left[\left(\frac{1}{\varsigma} - \frac{1}{3} + \frac{\lambda c_0 c_3}{3\varsigma} \varepsilon^{1-\mu+\tau} \right) \right. \\ & \left. \times c_1 \varepsilon^{3-2\mu} + \left(1 - \frac{1}{3\varsigma} - \frac{\lambda c_0 c_3}{3\varsigma} \varepsilon^{1-\mu+\tau} \right) \frac{c_2}{\varsigma^2} \varepsilon^{3-3\mu+\nu} \right] \end{aligned}$$

$$+ \eta \left[1 + \frac{\lambda c_0 c_3}{2} \left(1 - \frac{1}{\varsigma} \right) \varepsilon^{1-\mu+\tau} - \frac{\lambda^2 c_0^2 c_3^2}{6\varsigma} \varepsilon^{2(1-\mu+\tau)} \right] + \dots, \quad (31)$$

$$\begin{aligned} \left(\cos \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \right)^{-1} \Lambda_{21}(Q_1 \cup Q_5) &= \sqrt{\frac{\lambda c_0}{c_2}} \left[\sqrt{\lambda c_0 c_2} - \left(1 + \lambda c_0 c_3 \varepsilon^{1-\mu+\tau} \right) \right. \\ &\times \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \left. \varepsilon^{1-\mu} + \frac{\lambda c_0 c_1}{2} \sqrt{\frac{\lambda c_0}{c_2}} \left[\frac{\sqrt{\lambda c_0 c_2}}{3} - \left(1 + \frac{\lambda c_0 c_3}{3} \varepsilon^{1-\mu+\tau} \right) \right. \right. \\ &\times \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \left. \varepsilon^{3-2\mu} + \eta \left[1 + \frac{\lambda c_0 c_3}{2} \varepsilon^{1-\mu+\tau} \right. \right. \\ &\left. \left. - \frac{c_3}{2} \sqrt{\frac{\lambda c_0}{c_2}} \left(1 + \frac{\lambda c_0 c_3}{3} \varepsilon^{1-\mu+\tau} \right) \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \varepsilon^{1-\mu+\tau} \right] + \dots, \right. \end{aligned} \quad (32)$$

$$\begin{aligned} \left(\cosh \sqrt{\lambda c_0 c_1} \right)^{-1} \Lambda_{21}(Q_2 \cup Q_6) &= \sqrt{\frac{\lambda c_0}{c_1}} \left[\left(1 - \frac{\lambda c_0 c_3}{\varsigma} \varepsilon^{\tau-1} \right) \right. \\ &\times \tanh \sqrt{\lambda c_0 c_1} - \frac{\sqrt{\lambda c_0 c_1}}{\varsigma} \left. \varepsilon^{-1} + \frac{\lambda c_0 c_2}{2\varsigma^2} \left[\frac{\lambda c_0}{3\varsigma} \right. \right. \\ &\left. \left. - \sqrt{\frac{\lambda c_0}{c_1}} \left(1 - \frac{\lambda c_0 c_3}{3\varsigma} \varepsilon^{\tau-1} \right) \tanh \sqrt{\lambda c_0 c_1} \right] \varepsilon^{\nu-3} + \eta \left[1 - \frac{\lambda c_0 c_3}{2\varsigma} \varepsilon^{\tau-1} \right. \right. \\ &\left. \left. + \frac{c_3}{2} \sqrt{\frac{\lambda c_0}{c_1}} \left(1 - \frac{\lambda c_0 c_3}{3\varsigma} \varepsilon^{\tau-1} \right) \tanh \sqrt{\lambda c_0 c_1} \varepsilon^{\tau-1} \right] + \dots, \right. \end{aligned} \quad (33)$$

$$\begin{aligned} \left(\cosh \sqrt{\lambda c_0 c_1} \cos \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \right)^{-1} \Lambda_{21}(Q_0 \cup Q_3) &= \sqrt{\frac{\lambda c_0}{c_1 c_2}} \left(\sqrt{c_2} \tanh \sqrt{\lambda c_0 c_1} \right. \\ &\left. - \sqrt{c_1} \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} - \sqrt{\lambda c_0} c_3 \tanh \sqrt{\lambda c_0 c_1} \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \varepsilon^{\tau-1} \right) \varepsilon^{-1} \\ &+ \eta \left[1 + \frac{c_3}{2} \sqrt{\frac{\lambda c_0}{c_1 c_2}} \left(\sqrt{c_2} \tanh \sqrt{\lambda c_0 c_1} - \sqrt{c_1} \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} \right) \varepsilon^{\tau-1} \right. \\ &\left. - \frac{\lambda c_0 c_3^2}{6\sqrt{c_1 c_2}} \tanh \sqrt{\lambda c_0 c_1} \tan \sqrt{\frac{\lambda c_0 c_2}{\varsigma}} \varepsilon^{2(\tau-1)} \right] + \dots, \end{aligned} \quad (34)$$

respectively. Each of these four series contains the group of three terms at the singularity $\varepsilon^{1-\mu}$. There are two ways of cancellation of divergences in this group. One of these can be performed in the $\tau = \mu - 1$ plane, where the Q_0 , Q_1 , Q_2 , Q_4 elements are found (see also table 3). All these three terms compose the $\Lambda_{21}^{(0)}$ group and participate in the cancellation. The other way occurs on the Q_3 , Q_5 , Q_6 elements being subsets of the $\tau \geq 2(\mu - 1)$ half-space (table 3). Here, the $\Lambda_{21}^{(0)}$ group consists of two terms which are to be canceled out.

The cancellation of divergences imposes the constraints in the form of transparency equations (hereafter also called T_j -equations). Using these equations in the next terms of the expansion, one finds the subsets (called hereafter *transparency sets* or T_j -sets)

Table 4. Transparency equations and subsets of $S_{\delta'}$ given on each element Q_j , $j = 0, 1, \dots, 6$.

Q_j	T_j -equations	T_j -sets
Q_0	$\frac{\sqrt{c_2} \tanh \sqrt{\lambda c_0 c_1}}{\sqrt{c_1} + \sqrt{\lambda c_0 c_3} \tanh \sqrt{\lambda c_0 c_1}} = \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma}$	$T_0 \doteq \{\mu = \nu = 2, \tau = 1\} = Q_0$
Q_1	$\tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} = \frac{\sqrt{\lambda c_0 c_2}}{1 + \lambda c_0 c_3}$	$T_1 \doteq \{1 < \mu \leq 3/2, \nu = 2(\mu - 1), \tau = \mu - 1\}$
Q_2	$\tanh \sqrt{\lambda c_0 c_1} = \frac{\sqrt{\lambda c_0 c_1}}{\varsigma - \lambda c_0 c_3}$	$T_2 \doteq \{\mu = 2, \nu \geq 3, \tau = 1\}$
Q_3	$\sqrt{c_1} \tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} = \sqrt{c_2} \tanh \sqrt{\lambda c_0 c_1}$	$T_3 \doteq \{\mu = \nu = 2, \tau \geq 2\}$
Q_4	$\varsigma(1 - \lambda) = 1$	$T_4 \doteq \{1 < \mu \leq 3/2, \nu \geq 3(\mu - 1), \tau = \mu - 1\}$
Q_5	$\tan \frac{\sqrt{\lambda c_0 c_2}}{\varsigma} = \sqrt{\lambda c_0 c_2}$	$T_5 \doteq \{1 < \mu \leq 3/2, \nu = 2(\mu - 1), \tau \geq 2(\mu - 1)\}$
Q_6	$\varsigma \tanh \sqrt{\lambda c_0 c_1} = \sqrt{\lambda c_0 c_1}$	$T_6 \doteq \{\mu = 2, \nu \geq 3, \tau \geq 2\}$

$T_j \subset Q_j$, $j = 0, 1, \dots, 6$, where the limit g is finite. The results of these calculations are summarized in table 4 for each element Q_j of the $S_{\delta'}$ surface and illustrated by figure 2. The calculation of the $\varepsilon \rightarrow 0$ limit of the other transfer matrix elements with taking into account the T_j -equations gives representation (27).

7. Reduced transparency equations and an inverse problem for non-zero transmission

Thus, a non-zero transmission occurs under the transparency equations listed in table 4 and the constraints on the parameters c_0, c_1, c_2, c_3 and ς given in table 2. Next, the direct way would be inserting these constraints into the transparency equations. However, the resulting equations are not sufficiently convenient for a further analysis. Therefore, we simplify them by imposing some relations that couple c_0, c_1, c_2, c_3 and ς and do not contradict the constraints from table 2. These relations are listed in table 5 where for convenience of the corresponding transparency equation an additional parameter b has been incorporated. In this table, the interval $(0, \infty)$ means that the corresponding parameter is positive and arbitrary; the dependence $b(\lambda; \varsigma)$ or $c_0(\lambda; \varsigma)$ denotes that b or c_0 is a solution of the corresponding transparency equation. The list of reduced T_j -equations together with simplified expressions of χ and g is given in tables 6 and 7. As follows from table 7, for the case with $\eta = 0$ the effective δ -interaction appears only on the boundaries of the T_j -sets which are denoted by B_j , $j = 1, 2, \dots, 6$. These boundary sets are listed in table 8 and illustrated by figure 2. The appearance of

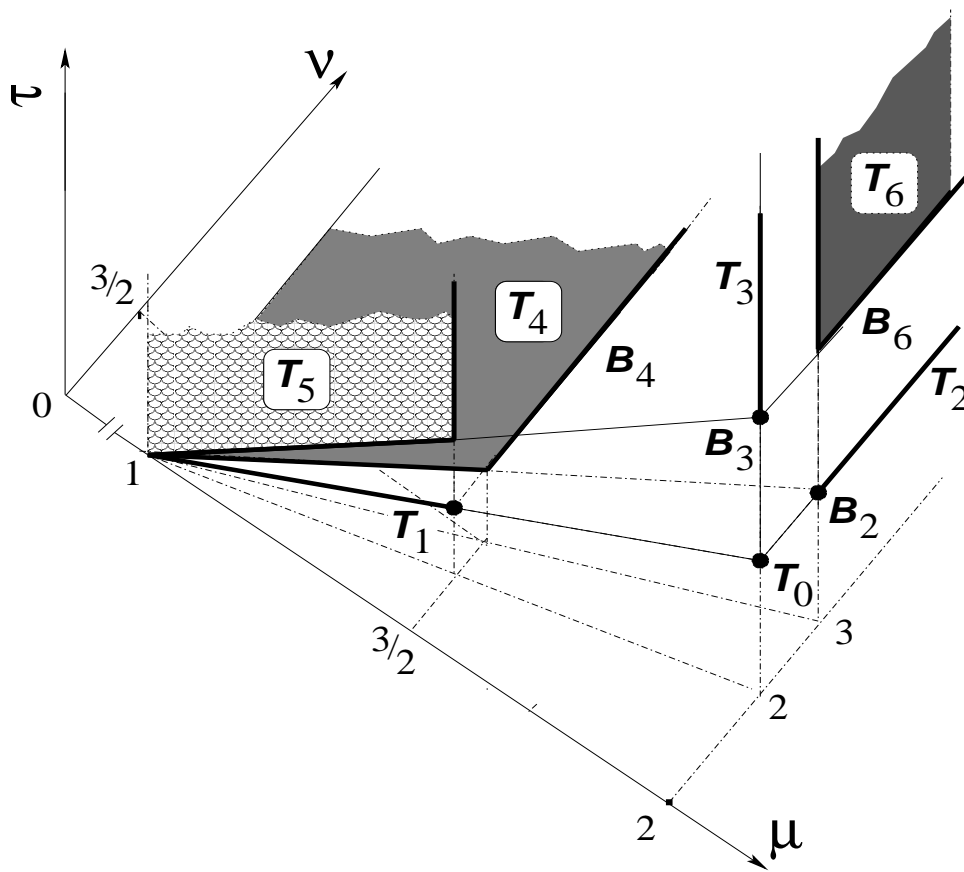


Figure 2. T_j -sets, $j = 0$ (point), $j = 1, 2, 3$ (lines) and $j = 4, 5, 6$ (planes), together with their boundaries B_j , $j = 1, 2, 3$ (points) and $j = 4, 5, 6$ (lines) on the trihedral surface $S_{\delta'}$. Notations for B_1 and B_5 are omitted. Sets B_1, B_2, B_3 are shown by balls and sets B_4, B_5, B_6 by thick lines.

Table 5. Coupling between constants b , c_0 , c_1 , c_2 , c_3 and ς imposed for simplification of transparency equations.

T_j	ς	b	c_0	c_1	c_2	c_3
T_0	$(0, \infty)$	$-$	$c_0(\lambda; \varsigma)$	$\frac{1}{1+c_0}$	$\frac{\varsigma^2}{1+c_0}$	$\frac{\varsigma}{c_0(1+c_0)}$
T_1	$(0, \infty)$	$-$	$c_0(\lambda; \varsigma)$	$(0, \infty)$	$\frac{2\varsigma^2}{1+c_0}$	$\frac{\varsigma}{c_0(1+c_0)}$
T_2	$(0, \infty)$	$-$	$c_0(\lambda; \varsigma)$	$\frac{2}{1+c_0}$	$(0, \infty)$	$\frac{\varsigma}{c_0(1+c_0)}$
T_3	$(0, \infty)$	$b(\lambda; \varsigma)$	$\frac{2}{(1+b^{-1})c_1}$	$(0, \infty)$	$\frac{c_1}{b}\varsigma^2$	$(0, \infty)$
T_4	$\frac{1}{1-\lambda}$	$-$	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$	$\frac{1}{(1-\lambda)c_0}$
T_5	$\frac{\tan\sqrt{2\lambda}}{\sqrt{2\lambda}}$	$-$	$(0, \infty)$	$(0, \infty)$	$\frac{\tan^2\sqrt{2\lambda}}{\lambda c_0}$	$(0, \infty)$
T_6	$\frac{\sqrt{2\lambda}}{\tanh\sqrt{2\lambda}}$	$-$	$(0, \infty)$	$2/c_0$	$(0, \infty)$	$(0, \infty)$

the δ -interaction on the B_j -sets is similar to that on the S_δ surface being the transition region from full to zero transmission.

As follows from table 6, for continuous test functions ($\varsigma = 1$) the T_0 -equation is

Table 6. Reduced transparency equations and corresponding values of χ for connection matrix (27).

T_j	Reduced T_j -equations	χ
T_0	$\frac{\tanh \sqrt{\frac{\lambda c_0}{1+c_0}}}{\frac{1}{\varsigma} + \sqrt{\frac{\lambda}{c_0(1+c_0)}} \tanh \sqrt{\frac{\lambda c_0}{1+c_0}}} = \tan \sqrt{\frac{\lambda c_0}{1+c_0}}$	$\frac{\varsigma \sinh \sqrt{\frac{\lambda c_0}{1+c_0}}}{\sin \sqrt{\frac{\lambda c_0}{1+c_0}}}$
T_1	$\left(\frac{1}{\varsigma} + \frac{\lambda}{1+c_0}\right) \tan \sqrt{\frac{2\lambda c_0}{1+c_0}} = \sqrt{\frac{2\lambda c_0}{1+c_0}}$	$\frac{1 + \frac{\lambda \varsigma}{1+c_0}}{\cos \sqrt{\frac{2\lambda c_0}{1+c_0}}}$
T_2	$\varsigma \left(1 - \frac{\lambda}{1+c_0}\right) \tanh \sqrt{\frac{2\lambda c_0}{1+c_0}} = \sqrt{\frac{2\lambda c_0}{1+c_0}}$	$\frac{\cosh \sqrt{\frac{2\lambda c_0}{1+c_0}}}{1 - \lambda/(1+c_0)}$
T_3	$\frac{\varsigma}{\sqrt{b}} \tanh \sqrt{\frac{2\lambda}{1+b^{-1}}} = \tan \sqrt{\frac{2\lambda}{1+b}}$	$\frac{\cosh \sqrt{\frac{2\lambda}{1+b^{-1}}}}{\cos \sqrt{\frac{2\lambda}{1+b}}}$
T_4	$\varsigma(1 - \lambda) = 1$	$\frac{1}{1-\lambda}$
T_5	$\tan \sqrt{2\lambda} = \varsigma \sqrt{2\lambda}$	$\left(\cos \sqrt{2\lambda}\right)^{-1}$
T_6	$\varsigma \tanh \sqrt{2\lambda} = \sqrt{2\lambda}$	$\cosh \sqrt{2\lambda}$

Table 7. Values of g for connection matrix (27) calculated for each T_j -set. Here $\delta_{\alpha,\beta}$ stands for the Kronecker delta symbol.

T_j	g
T_0	$\eta \left[\cos \sqrt{\frac{\lambda c_0}{1+c_0}} \cosh \sqrt{\frac{\lambda c_0}{1+c_0}} + \frac{\lambda \varsigma}{3c_0(1+c_0)} \sin \sqrt{\frac{\lambda c_0}{1+c_0}} \sinh \sqrt{\frac{\lambda c_0}{1+c_0}} \right]$
T_1	$\left(1 + \frac{\lambda \varsigma}{1+c_0}\right)^{-1} \left[\eta \left(1 + \frac{\lambda^2 \varsigma^2}{3(1+c_0)^2}\right) - \frac{\lambda^2 c_0^2 c_1}{3} \delta_{\mu,3/2} \right] \cos \sqrt{\frac{2\lambda c_0}{1+c_0}}$
T_2	$\left(1 - \frac{\lambda}{1+c_0}\right)^{-1} \left[\eta \left(1 - \frac{\lambda}{1+c_0} + \frac{\lambda^2}{3(1+c_0)^2}\right) - \frac{\lambda^2 c_0}{3\varsigma^2(1+c_0)} \delta_{\nu,3} \right] \cosh \sqrt{\frac{2\lambda c_0}{1+c_0}}$
T_3	$\left[\eta - \frac{2\lambda c_3}{(1+b^{-1})c_1^2} \tanh^2 \sqrt{\frac{2\lambda}{1+b^{-1}}} \delta_{\tau,2} \right] \cos \sqrt{\frac{2\lambda}{1+b}} \cosh \sqrt{\frac{2\lambda}{1+b^{-1}}}$
T_4	$\eta \left[1 + \frac{\lambda^2}{3(1-\lambda)} \right] - \frac{\lambda^2 c_0^2}{3} (1-\lambda) [c_1 \delta_{\mu,3/2} + c_2 (1-\lambda) \delta_{\nu,3(\mu-1)}]$
T_5	$\left[\eta - \lambda^2 c_0^2 \left(\frac{c_1}{3} \delta_{\mu,3/2} + c_3 \delta_{\tau,2(\mu-1)} \right) \right] \cos \sqrt{2\lambda}$
T_6	$\left[\eta - \frac{\lambda c_0^2}{2} \left(\frac{c_2}{3\sqrt{2\lambda}} \tanh \sqrt{2\lambda} \delta_{\nu,3} + c_3 \delta_{\tau,2} \right) \tanh^2 \sqrt{2\lambda} \right] \cosh \sqrt{2\lambda}$

invariant under the transformation $\lambda \rightarrow -\lambda$. With this transformation the elements in the pairs (T_1, T_2) and (T_5, T_6) are transposed. Next, the T_3 -equation is invariant under the transformation $\lambda \rightarrow -\lambda$, $b \rightarrow b^{-1}$. In all these cases, we obtain the replacement $\chi \rightarrow \chi^{-1}$. The reduced apex and edge T_j -equations (with $j = 0, 1, 2, 3$) contain the two parameters ς and c_0 , whereas the plane T_j -equations (with $j = 4, 5, 6$) only the parameter ς . Therefore, in the former case one can tune two parameters to construct a regularizing sequence for a given λ . For instance, fixing ς , i.e., a corresponding space of test functions \mathcal{D}_ς , one can find c_0 as a function of λ . In the latter case, for each available λ we have to fix a space \mathcal{D}_ς according to the solution of the plane T_j -equations. These solutions $\varsigma = \varsigma(\lambda)$ are trivial (see the last three lines in table 6).

Finally, for any given $\varsigma > 0$, the apex and edge T_j -equations can be solved numerically. The numerical solution of these equations for the case of continuous test functions ($\varsigma = 1$) and $\lambda \in (0, \infty)$ is present in figure 3 where $j = 0, 1, 3$. Here, for any λ exceeding some critical value λ_c , there exists a countable sets of roots. Having the solutions for λ , one can calculate the matrix elements χ and g according to the equations listed in tables 6 and 7, respectively.

8. Reflection-transmission coefficients and bound states

Having the values for χ and g (see tables 6 and 7), one can calculate the reflection-transmission coefficients and bound states through the elements of the matrix Λ in the standard way [27, 21, 26]. Indeed, using the definition for the reflection and transmission coefficients according to the equations

$$\psi(x) = \begin{cases} e^{ikx} + R e^{-ikx} & \text{for } -\infty < x < x_1, \\ T e^{ikx} & \text{for } x_2 < x < \infty, \end{cases}, \quad (35)$$

one can rewrite the following equations

$$\begin{pmatrix} T \\ ikT \end{pmatrix} e^{-ik(x_1-x_2)} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} 1 + R \\ ik(1 - R) \end{pmatrix}. \quad (36)$$

Solving next this matrix equation with respect to the coefficients R and T , one finds their representation in terms of the matrix elements Λ_{ij} :

$$R = -\frac{\Lambda_{11} - \Lambda_{22} + i(k\Lambda_{12} + k^{-1}\Lambda_{21})}{\Delta} \quad \text{and} \quad T = \frac{2}{\Delta} e^{-ikx_0} \quad (37)$$

where $\Delta \doteq \Lambda_{11} + \Lambda_{22} - i(k\Lambda_{12} - k^{-1}\Lambda_{21})$. One can easily check the validity of the conservation law $|R|^2 + |T|^2 = 1$.

Next we denote

$$u \doteq \Lambda_{11} - \Lambda_{22} \quad \text{and} \quad v \doteq k\Lambda_{12} + k^{-1}\Lambda_{21}. \quad (38)$$

Then, we obtain with the use of the equality $\Lambda_{11}\Lambda_{22} - \Lambda_{12}\Lambda_{21} = 1$ the following basic equations for the reflectibility and transmissibility:

$$|R|^2 = \frac{u^2 + v^2}{4 + u^2 + v^2} \quad \text{and} \quad |T|^2 = \frac{4}{4 + u^2 + v^2}. \quad (39)$$

Table 8. Boundary sets $B_j \subset T_j$, $j = 1, 2, \dots, 6$: points B_1, B_2, B_3 and lines B_4, B_5, B_6 .

B_j -sets
$B_1 \doteq \{\mu = 3/2, \nu = 1, \tau = 1/2\}$
$B_2 \doteq \{\mu = 2, \nu = 3, \tau = 1\}$
$B_3 \doteq \{\mu = \nu = \tau = 2\}$
$B_4 \doteq \{1 < \mu \leq 3/2, \nu = 3(\mu - 1), \tau = \mu - 1\} \cup \{\mu = 3/2, \nu > 3/2, \tau = 1/2\}$
$B_5 \doteq \{1 < \mu \leq 3/2, \nu = 2(\mu - 1), \tau = 2(\mu - 1)\} \cup \{\mu = 3/2, \nu = 1, \tau > 1\}$
$B_6 \doteq \{\mu = 2, \nu \geq 3, \tau = 2\} \cup \{\mu = 2, \nu = 3, \tau > 2\}$

These coefficients can be rewritten in the form

$$|R|^2 = \frac{(\chi - \chi^{-1})^2 + g^2/k^2}{4 + (\chi - \chi^{-1})^2 + g^2/k^2}, \quad |T|^2 = \frac{4}{4 + (\chi - \chi^{-1})^2 + g^2/k^2}. \quad (40)$$

In the case when $g \neq 0$, i.e. a δ -potential is present, one can expect the existence of a nontrivial bound state with energy $E \doteq -\kappa^2$. Indeed, looking for negative-energy solutions of equation (1) in the form

$$\psi(x) = \begin{cases} A e^{\kappa x} & \text{for } -\infty < x < 0, \\ B e^{-\kappa x} & \text{for } 0 < x < \infty, \end{cases}, \quad (41)$$

one can write the matrix equation

$$\begin{pmatrix} B \\ -\kappa B \end{pmatrix} = \begin{pmatrix} \chi & 0 \\ g & \chi^{-1} \end{pmatrix} \begin{pmatrix} A \\ \kappa A \end{pmatrix}. \quad (42)$$

The compatibility of solutions to this equation gives the equation for κ from which we obtain

$$\kappa = -\frac{g}{\chi + \chi^{-1}}. \quad (43)$$

Thus, for a given λ one can also calculate the bound level κ .

As three examples, we have checked the inverse problem solution for the T_0 -, T_1 - and T_3 -sets using the transfer matrix Λ given by equations (24) with sufficiently small ε . To this end, we solve numerically the corresponding T_j -equations from table 6 with respect to c_0 and b , respectively, and thus obtain the multivalued functions $c_0 = c_0(\lambda)$ and $b = b(\lambda)$. Next we fix any values $\lambda = \bar{\lambda}$ on each T_0 -, T_1 - and T_3 -sets. It is sufficient to use only the first roots of the T_j -equations. Inserting the values of $c_0(\bar{\lambda})$ and $b(\bar{\lambda})$ (see figure 3) for given $\bar{\lambda}$'s into (18) and (25) and using table 5, we obtain the corresponding expressions for the parameters l, r, ρ, p, q, s , which are summarized in table 9. Here, the positive parameters c_1 and c_3 are arbitrary. Finally, inserting these parameters into equations (24), (38) and (39), we plot the resonance behaviour of $|T|^2$ as a function of λ as illustrated by figure 4. As shown in this figure (see the red lines), the given $\bar{\lambda}$'s belong to the resonance sets T_0, T_1 and T_3 .

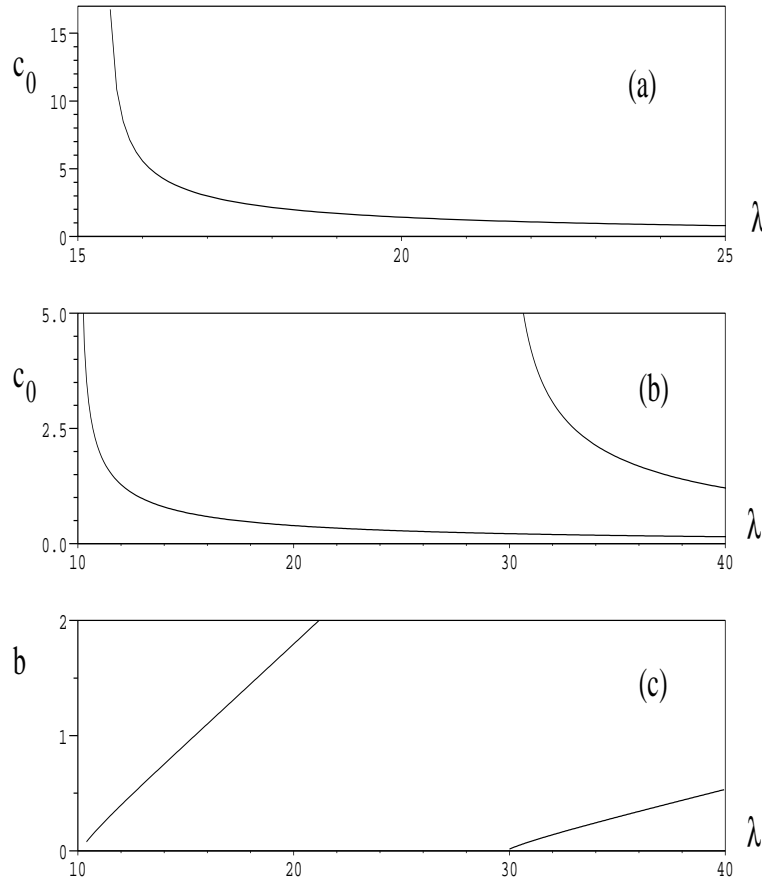


Figure 3. Numerical solutions of the reduced T_j -equations (see table 6) with $j = 0, 1, 3$ for continuous test functions ($\varsigma = 1$): (a), (b) $c_0 = c_0(\lambda)$ and (c) $b = b(\lambda)$. The only first-root λ -dependence is plotted in (a) and the two first-root dependencies are plotted in (b) and (c).

9. Conclusions and discussion

In this paper, we have used a multi-parametric family of regularizing sequences $\Delta'_\varepsilon(x)$ of the δ' -like shape (barrier-well rectangles with a squeezing parameter ε) to approximate both Dirac's delta function $\delta(x)$ and its derivative $\delta'(x)$. A set of approximating parameters allows us to use a 'non-uniform squeezing' of rectangles instead of the standard 'uniform squeezing' defined by the regularization (2) with a dipole-like function $\mathcal{V}(\xi)$. In our approach, we have incorporated three powers μ, ν, τ and found in the $\{\mu, \nu, \tau\}$ -space the two trihedral surfaces S_δ and $S_{\delta'}$ (illustrated by figure 1) that correspond to the distributional limits $\Delta'_\varepsilon(x) \rightarrow \delta(x)$ and $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ (as $\varepsilon \rightarrow 0$), respectively. Each triple point on S_δ or $S_{\delta'}$ determines a pathway along which the $\delta(x)$ or $\delta'(x)$ function is obtained. On the one hand, this regularization procedure generalizes pathway (2), but on the other hand, it narrows the general class of dipole-like functions to the family of only piecewise functions. Since the limit $\Delta'_\varepsilon(x) \rightarrow \delta(x)$ can be realized,

Table 9. Parameters l, r, ρ, p, q, s used for calculation of transmissibility $|T|^2$ according to equations (24), (38) and (39). Here parameters $c_0 = c_0(\bar{\lambda})$ and $b = b(\bar{\lambda})$ are fixed for a given $\bar{\lambda}$. Parameters c_1 and c_3 are arbitrary.

T_j	l	r	ρ
T_0	$\frac{\varepsilon}{1+c_0}$	$\frac{\varepsilon}{1+c_0}$	$\frac{\varepsilon}{c_0(1+c_0)}$
T_1	$c_1\varepsilon$	$\frac{2}{1+c_0}\varepsilon^{\mu-1}, \quad \mu \leq 3/2$	$\frac{1}{c_0(1+c_0)}\varepsilon^{\mu-1}, \quad \mu \leq 3/2$
T_3	$c_1\varepsilon$	$\frac{c_1}{b}\varepsilon$	$c_3\varepsilon^\tau, \quad \tau \geq 2$
T_j	p	q	s
T_0	$i\sqrt{\lambda c_0(1+c_0)\varepsilon^{-2} - k^2}$	$\sqrt{\lambda c_0(1+c_0)\varepsilon^{-2} + k^2}$	$i\sqrt{\eta c_0(1+c_0)\varepsilon^{-1} - k^2}$
T_1	$i\sqrt{\lambda(c_0/c_1)\varepsilon^{-\mu} - k^2}$	$\sqrt{\lambda(c_0/2)(1+c_0)\varepsilon^{2(1-\mu)} + k^2}$	$i\sqrt{\eta c_0(1+c_0)\varepsilon^{1-\mu} - k^2}$
T_3	$i\sqrt{\frac{2\bar{\lambda}b}{1+b}c_1^{-2}\varepsilon^{-2} - k^2}$	$\sqrt{\frac{2\bar{\lambda}b^2}{1+b}c_1^{-2}\varepsilon^{-2} + k^2}$	$i\sqrt{\frac{\eta}{c_3}\varepsilon^{-\tau} - k^2}$

it is worth to add in the Schrödinger equation (1) the δ -potential with the corresponding approximation $\Delta_\varepsilon(x) \rightarrow \delta(x)$.

Having explicit finite-range solutions of the Schrödinger equation (1) given through the transfer matrix (23)-(25), one can control the cancellation of divergences that appear in the kinetic and potential energy terms along each pathway. This cancellation takes place for both the $\varepsilon \rightarrow 0$ limits: $\Delta'_\varepsilon(x) \rightarrow \delta(x)$ and $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$. In the δ -case, the Schrödinger operator is well defined on the whole surface S_δ , whereas in the second case, it is defined only on some subsets of $S_{\delta'}$. This means that the existence or non-existence of the transparency regime depends on the pathway $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$. In particular, the different but correct scattering results: (i) the existence of resonance sets when regularizing by limit (2) (proved in [18, 19, 28]) and (ii) and the zero transmission when using limit (4) with $\alpha = 1$ (calculated in [29]) can now be understood. It should be noticed here that the similar cancellation procedure has been realized to construct δ -like point interactions in two [30] and three [31] dimensions as well as in the quantum field theory for the construction of non-trivial Hamiltonians for the Yukawa $_{1+1}$ and ϕ^4_{2+1} interactions (for more details see [32] and the comments with extensive bibliography on the constructive field theory given in [33]). The similar problem arises also in the relativistic scattering by the δ -potential [35]. A three particle system with δ -potentials in two and three dimensions has been studied in [34].

The main goal of this paper was to solve the following inverse problem: For a given

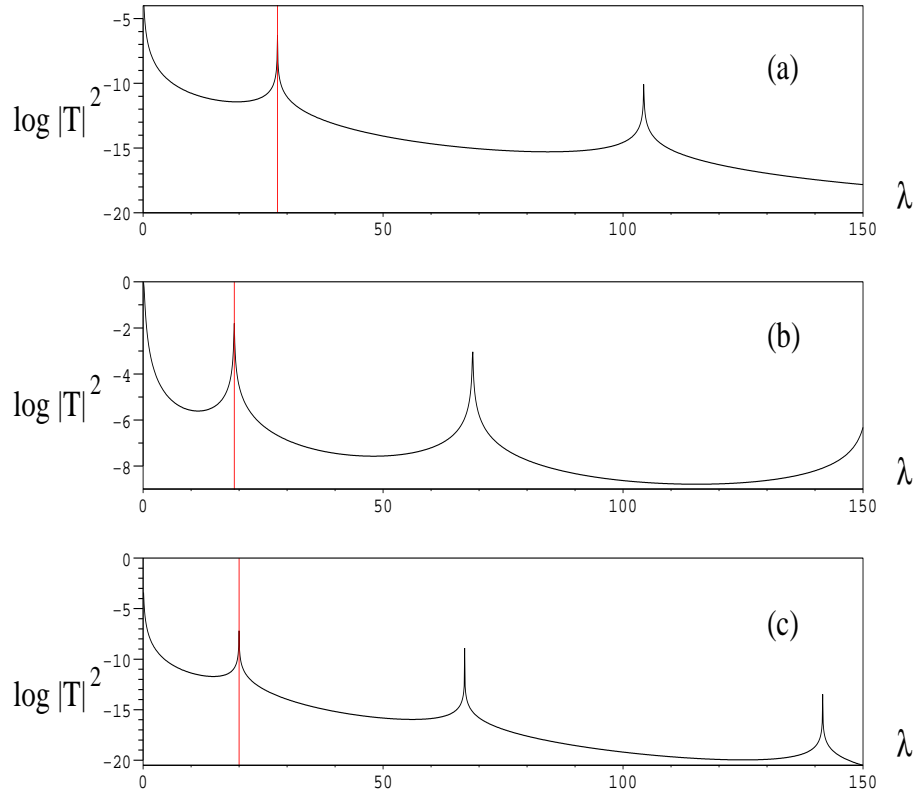


Figure 4. (Color online) Resonant transmission for a given $\lambda = \bar{\lambda}$ (shown by the vertical red lines) for three transparency sets T_0 , T_1 and T_3 calculated according to (24), (38) and (39) with parameters given in table 9: (a) $T_0 = Q_0$, $\bar{\lambda} = 28$, $c_0 = c_0(\bar{\lambda})$; (b) $T_1 = B_1$, $\bar{\lambda} = 19$, $c_0 = c_0(\bar{\lambda})$ and (c) $T_3 = B_3$, $\bar{\lambda} = 20$, $b = b(\bar{\lambda})$. In all cases $\varsigma = 1$, $\eta = 0$, $k = 1$, $c_1 = c_3 = 1$ and $\varepsilon = 0.0001$.

$\bar{\lambda} \in \mathbb{R} \setminus 0$ to construct a regularizing sequence $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ as $\varepsilon \rightarrow 0$ such that $\bar{\lambda}$ would belong at least to one of the transparency sets T_j 's, $j = 0, 1, \dots, 6$. To this end, we have developed an approach of constructing the family of regularizing sequences that contains free parameters as many as possible. In the δ' -limit (when $a_1 = a_2 = a_3 = 0$), these are the positive constants ς ; c_0 , c_1 , c_2 , c_3 with some constraints that provide the limiting distribution $\delta'(x)$. As illustrated by table 4, on each transparency set T_j , $j = 0, 1, \dots, 6$, the T_j -equation for resonances has the same form depending only on these constants. Taking into account the constraints given in table 2, the number of these constants can be reduced and one of the ways of this reduction together with an appropriate simplification is present in table 5. The resulting simplified T_j -equations and the matrix elements χ and g are given by tables 6 and 7. On the sets T_0 , T_1 , T_2 , T_3 , except for $\varsigma > 0$, it is possible to get out one parameter (c_0 or b) which can be tuned at fixed ς in each of the corresponding reduced T_j -equations. Solving these equations at a given λ , one can find numerically the dependence $c_0 = c_0(\lambda; \varsigma)$ or $b = b(\lambda; \varsigma)$. If we fix $\varsigma = 1$, the inverse problem can be solved for the space of continuous test functions. However,

it is impossible to have such a parameter in the transparency equations on the sets T_4, T_5, T_6 . Here, for a given λ we have to fix ς , i.e. the space of discontinuous test functions \mathcal{D}_ς .

Finally, note that in spite of the Schrödinger operators given by equation (1) and regularized by the sequence $\Delta'_\varepsilon(x) \rightarrow \delta(x)$ or $\Delta'_\varepsilon(x) \rightarrow \delta'(x)$ are well defined in the $\varepsilon \rightarrow 0$ limit, the corresponding point interaction models are ambiguous. From a physical point of view this means that in some cases the exact description of a concrete point interaction model has to be accompanied by a regularizing sequence.

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